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EFFECTIVENESS STUDIES OF MISSILE SYSTEMS AGAINST GROUND TARGET

Part II Spherical Damage Patterns vs. Point Targets

(a) Mathematical Formulation (U)

by

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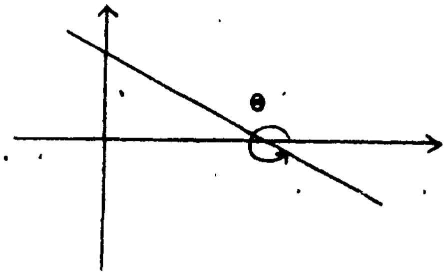
INTRODUCTION: In this section, the effectiveness of a warhead having a spherical damage pattern is evaluated against a point target on the ground. Many of the formulae previously obtained simplify to such extent that a separate treatment seems justified.

We shall assume that the warhead is carried in a missile whose trajectory, at least that part of interest to our problem, is a straight line making an angle θ with the direction of increasing range, and that the velocity during this terminal phase is constant.



FORMULATION OF THE PROBLEM: The point target is placed at the origin of the (x, y, z) coordinate system, where the (x, y) plane is the plane of the earth and z represents altitude above the ground. If there were no lateral dispersion, the trajectory would lie in the (x, z) plane. If a warhead having a damage radius s, bursts at a point (x, y, z), when $z \geq 0$ (the burst occurs on or above the ground), the damage effectiveness sphere contains the target in its interior or on its boundary if and only if $x^2 + y^2 + z^2 \leq s^2$. When $z < 0$, the missile would have hit the ground at the point $(x - z \cot \theta, y, 0)$ and we shall assume that the warhead detonated at that point. Thus, when $z < 0$, the target lies on or within the damage effectiveness sphere when $(x - z \cot \theta)^2 + y^2 \leq s^2$.

Since we identify the burst point with the center of the damage effectiveness sphere, the probability that the warhead bursts at the point (x, y, z) becomes the probability that the center of the damage effectiveness sphere is at the point (x, y, z). If the probability density function for this occurrence is denoted by $p(x, y, z)$, then P, the probability that the target lies within or on a damage effectiveness sphere associated with a given burst, is

$$P = P_1 + P_2$$

where

$$P_1 = \iiint\limits_{z \geq ;\ x^2 + y^2 + z^2 \leq s^2} p(x, y, z)\, dV_{xyz}$$

$$P_2 = \iiint\limits_{z < 0;\ (x - z \cot \theta)^2 + y^2 \leq s^2} p(x, y, z)\, dV_{xyz}$$

Example: In what follows, we shall assume that $p(x, y, z)$ is a tri-variate normal distribution with mean $(\bar{x}, 0, \bar{z})$, standard deviations σ_x, σ_y, σ_z and correlation $\rho = \rho_{xz}$. We assume $\rho_{xy} = \rho_{yz} = 0$.

Thus

$$p(x,y,z) = \frac{1}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\bar{x}}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\bar{x}}{\sigma_x}\right)\left(\frac{z-\bar{z}}{\sigma_z}\right) + \left(\frac{z-\bar{z}}{\sigma_z}\right)^2\right] - \frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2}$$

Since p(x, y, z) is an even function of y,

$$P_1 = \frac{2}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z\sqrt{1-\rho^2}} \int_{-S}^{S} dx \int_0^{\sqrt{S^2-x^2}} dz \int_0^{\sqrt{S^2-x^2-z^2}} e^{-\frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2 - \frac{G_1}{2(1-\rho^2)}} dy$$

where $G_1 = \left[\left(\frac{x-\bar{x}}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\bar{x}}{\sigma_x}\right)\left(\frac{z-\bar{z}}{\sigma_z}\right) + \left(\frac{z-\bar{z}}{\sigma_z}\right)^2\right]$

If we denote

$$\alpha(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt$$

then

$$P_1 = \frac{2}{2\pi\,\sigma_x\,\sigma_z\sqrt{1-\rho^2}} \int_{-S}^{S} dx \int_0^{\sqrt{S^2-x^2}} \alpha\left(\frac{\sqrt{S^2-x^2-z^2}}{\sigma_y}\right) e^{-\frac{G_1}{2(1-\rho^2)}} dz$$

The integration in the (x, z) plane is now over the half circle $x^2 + z^2 \leq S^2$; $z \geq 0$. This suggests the transformation

$$x = s \sin\beta \cos\alpha \qquad 0 \le \beta \le \pi/2$$
$$z = s \sin\beta \sin\alpha \qquad 0 \le \alpha \le \pi$$

Under this transformation

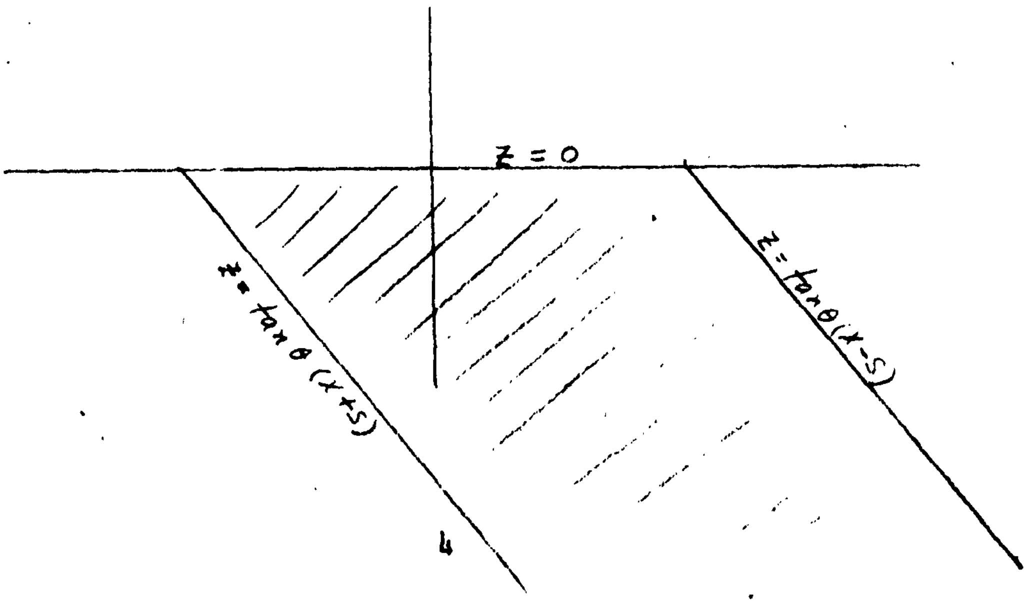
$$P_1 = \frac{1}{\pi\sqrt{1-\rho^2}}\left(\frac{s}{\sigma_x}\right)\left(\frac{s}{\sigma_z}\right)\int_0^{\frac{\pi}{2}} d\beta \int_0^{\pi} \sin\beta \cos\beta \; \mathcal{O}\!\left(\frac{s}{\sigma_y}\cos\beta\right) e^{-\frac{G_2}{2(1-\rho^2)}} d\alpha$$

where

$$G_2 = \left(\frac{s\sin\beta\cos\alpha - \bar{x}}{\sigma_x}\right)^2 - 2\rho\left(\frac{s\sin\beta\cos\alpha - \bar{x}}{\sigma_x}\right)\left(\frac{s\sin\beta\sin\alpha - \bar{z}}{\sigma_z}\right) + \left(\frac{s\sin\beta\sin\alpha - \bar{z}}{\sigma_z}\right)^2$$

This representation for P_1 is used since it is easy to program for machine computations.

We proceed in the following fashion to compute P_2. Again, we use the fact that $p(x, y, z)$ is an even function of y. If the integration with respect to y is performed first, the remaining integrations with respect to x and z will be over the region defined by the x axis, and the lines $z = \tan\theta\,(x \pm s)$



$$P_2 = \frac{2}{(2\pi)\,\sigma_x \sigma_z \sqrt{1-\rho^2}} \int_{-\infty}^{0} dz \int_{z\cot\theta - s}^{z\cot\theta + s} \mathcal{A}\left(\frac{\sqrt{s^2-(x-z\cot\theta)^2}}{\sigma_y}\right) e^{-\frac{G_1}{2(1-\rho^2)}}\, dx$$

where G_1 has been previously defined.

For computational purposes, the transformation of coordinates

$$x = s(\alpha + \beta \cot\theta)$$

$$z = s\beta$$

produces

$$P_2 = \frac{1}{\pi\sqrt{1-\rho^2}}\left(\frac{s}{\sigma_x}\right)\left(\frac{s}{\sigma_z}\right) \int_{-\infty}^{0} d\beta \int_{-1}^{+1} \mathcal{A}\left(\frac{s}{\sigma_y}\sqrt{1-\alpha^2}\right) e^{-\frac{G_3}{2(1-\rho^2)}}\, d\alpha$$

where

$$G_3 = \left[\left(\frac{s}{\sigma_x}\alpha + \left(\frac{s}{\sigma_x}\cot\theta\right)\beta - \frac{\bar{x}}{\sigma_x}\right)^2 - 2\rho\left(\frac{s}{\sigma_x}\alpha + \frac{s}{\sigma_x}\cot\theta\,\beta - \frac{\bar{x}}{\sigma_x}\right)\left(\frac{s}{\sigma_z}\beta - \frac{\bar{z}}{\sigma_z}\right) + \left(\frac{s}{\sigma_z}\beta - \frac{\bar{z}}{\sigma_z}\right)^2\right].$$

At times it is desirable to determine the weapon effectiveness when perfect fusing is assumed. For perfect altitude fuzing, $\sigma_z = 0$, $z = \bar{z}$ and the preceding formulae require modification. The probability P_2 of a ground burst is zero and the probability density function becomes the bivariate normal distribution:

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\, e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma_y}\right)^2 - \frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2}$$

The probability P that the target lies within or on a damage effectiveness sphere associated with a given burst is

$$P = \iint\limits_{x^2+y^2 \leq S^2-\bar{z}^2} p(x,y)\, dA_{xy} = \frac{1}{2\pi\sigma_x\sigma_y} \iint\limits_{x^2+y^2 \leq S^2-\bar{z}^2} e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma_x}\right)^2 - \frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2} dx\, dy$$

$$= \frac{2}{\sqrt{2\pi}\,\sigma_x} \int_{-\sqrt{S^2-\bar{z}^2}}^{\sqrt{S^2-\bar{z}^2}} a\left(\frac{\sqrt{S^2-\bar{z}^2-x^2}}{\sigma_y}\right) e^{-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma_x}\right)^2} dx$$

If we let

$$x = \sqrt{S^2-\bar{z}^2}\,\alpha$$

$$P = \frac{2}{\sqrt{2\pi}}\left(\frac{\sqrt{S^2-\bar{z}^2}}{\sigma_x}\right)\int_{-1}^{+1} a\left(\frac{\sqrt{S^2-\bar{z}^2}}{\sigma_y}\sqrt{1-\alpha^2}\right) e^{-\frac{1}{2}\left(\frac{\sqrt{S^2-\bar{z}^2}}{\sigma_x}\alpha - \frac{\bar{x}}{\sigma_x}\right)^2} d\alpha$$

which is the representation used for the computation.